1

Two samples of 50 observations each produce the following moment matrices. In each case, X is a constant and one variable.

	Sample 1	Sample 2
$X'X$	$\begin{bmatrix} 50 & 300 \\ 300 & 2100 \end{bmatrix}$	$\begin{bmatrix} 50 & 300 \\ 300 & 2100 \end{bmatrix}$
$y'X$	$\begin{bmatrix} 300 & 2000 \end{bmatrix}$	$\begin{bmatrix} 300 & 2200 \end{bmatrix}$
$y'y$	$\begin{bmatrix} 2100 \end{bmatrix}$	$\begin{bmatrix} 2800 \end{bmatrix}$

1.a

Compute the least squares regression coefficients and the residual variances σ^2 for each data set. Compute the R^2 for each regression.

$$b_1 = (X_1'X_1)^{-1}X_1'y_1 = \begin{bmatrix} 2 \\ \frac{2}{3} \end{bmatrix} \qquad \sigma_1^2 = \frac{y_1'y_1 - b_1X_1'y_1}{T-K} = 3.4722$$

$$R_1^2 = 0.4444$$

$$b_2 = (X_2'X_2)^{-2}X_2'y_2 = \begin{bmatrix} -2 \\ \frac{4}{3} \end{bmatrix} \qquad \sigma_2^2 = \frac{y_2'y_2 - b_2X_2'y_2}{T-K} = 9.7222$$

$$R_1^2 = 0.5333$$

1.b

Compute the least squares estimate of the coefficients assuming that the coefficients and disturbance variance are the same in the two regressions. Also compute the estimate of the variance error and the covariance matrix of the estimate.

$$X_a'X_a = X_1'X_1 + X_2'X_2 = \begin{bmatrix} 100 & 600 \\ 600 & 4200 \end{bmatrix}$$
$$X_a'y_a = X_1'y_1 + X_2'y_2 = \begin{bmatrix} 600 \\ 4200 \end{bmatrix}$$
$$y'y = 4900$$

$$b_a = (X_a'X_a)^{-1}X_a'y_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \sigma_a^2 = y_a'y_a - b_a'X_a'y_a = 7.1429$$

$$cov(b_a) = \sigma_a^2(X_a'X_a)^{-1} = \begin{bmatrix} 0.500000000000002 & -0.0714285714285718 \\ -0.0714285714285718 & 0.0119047619047620 \end{bmatrix}$$

1.c

Test the hypothesis that the variances in the two regressions are the same.

H_0: $\sigma_1^2 = \sigma_2^2$
H_1: $\sigma_1^2 \neq \sigma_2^2$

$$GQtest : \frac{\sigma^2_{\text{high}}}{\sigma^2_{\text{low}}} \tag{1}$$
$$= \frac{9.7222}{3.4722} \tag{2}$$
$$= 2.8 \tag{3}$$

The F_c at $\alpha = 0.05$ and 48 degrees of freedom is 1.6154, which means $F > F_c$ ($2.8 > 1.6154$). We have sufficient evidence to reject H_0 and conclude the variances are different.

1.d

Compute the FGLS estimator. What is the covariance matrix of the estimate. Compare it with the result of part b.

$$b_{FGLS} = (\frac{(X_1'X_1)}{\sigma_1^2} + \frac{(X_2'X_2)}{\sigma_2^2})^{-1}(\frac{X_1'y_1}{\sigma_1^2} + \frac{X_2'y_2}{\sigma_2^2}) \tag{1}$$
$$= \begin{bmatrix} 0.9474 \\ 0.84210 \end{bmatrix} \tag{2}$$

2 §15.3

Reconsider the household expenditure model that appears in the text, the data for which appear in Table 5.2. That is, we have the model

$$y_t = \beta_t + \beta_2 x_t + e_t$$

where y_t is food expenditure for the tth household and x_t is income.

2.a

Find generalized least squares estimates for β_1 and β_2 under the assumption that $var(e_t) = \sigma_t^2 = \sigma^2 x_t$

$$\hat{\beta} = (X*' X*)^{-1} X*' y* \tag{1}$$
$$= (X'V^{-1}X)^{-1} X'V^{-1}y \tag{2}$$

2.b

Now suppose

$$var(e_t) = \sigma_t^2 = \sigma^2 x_t^\gamma$$

where γ is an unknown parameter.

2.b.(i)

Show that we can write

$$\sigma_t^2 = exp\{\alpha + \gamma \ln x_t\} = \sigma^2 x_t^\gamma$$

where $\alpha = \ln \sigma^2$.

$$\sigma_t^2 = \sigma^2 x_t^{\gamma} \tag{1}$$
$$\ln(\sigma_t^2) = \ln(\sigma^2 x_t^{\gamma}) \tag{2}$$
$$\ln(\sigma_t^2) = \ln(\sigma^2) + \ln(x_t^{\gamma}) \tag{3}$$
$$\ln(\sigma_t^2) = \ln(\sigma^2) + \gamma \ln(x_t) \tag{4}$$
$$exp\{\ln(\sigma_t^2)\} = exp\{\ln(\sigma^2) + \gamma \ln(x_t)\} \tag{5}$$
$$\sigma_t^2 = exp\{\alpha + \gamma \ln(x_t)\} \tag{6}$$

2.b.(ii)

Find least squares estimates for β_1 and β_2 and the corresponding least squares residuals $(\hat{\epsilon}_t)$.

$$\hat{\beta} = \begin{bmatrix} -40.2498 \\ 15.1915 \end{bmatrix} \qquad \hat{\epsilon}_t = \begin{bmatrix} 0.0986857154845489 \\ 8.40256717697635 \\ 3.61271028284606 \\ 12.6123392205323 \\ 17.1315396209139 \\ 0.373602606901820 \\ 8.56791611934751 \\ 2.68270103709577 \\ 7.22900457517643 \\ 2.53308128047939 \\ 2.45085319550288 \\ 14.5938186781840 \\ 122.822764014769 \\ 9.36250301632493e-06 \\ 1.51214860350702 \\ 34.8647039088041 \\ 4.15359597209790 \\ 74.2878270745445 \\ 74.2699769919263 \\ 0.632128339019488 \\ 48.2310383493441 \\ 5.19419383629467 \\ 3.60061386839072 \\ 3.07115685695939 \\ 16.1596631457760 \\ 152.114136442314 \\ 12.9784717868131 \\ 3.21751065826876 \\ 17.4546401885470 \\ 50.5088858941612 \\ 14.3022453743355 \\ 201.095191109136 \\ 133.910081701055 \\ 111.121974159155 \\ 8.24517227314838 \\ 108.148659753797 \\ 72.5473957052534 \\ 0.0321336739440184 \\ 117.015297871053 \\ 282.799068942443 \end{bmatrix}$$

2.b.(iii)

Estimate α and γ through application of least squares to the equation

$$\ln \hat{e}_t^2 = \alpha + \gamma \ln x_t + v_t$$

where v_t is an error term. Using the standard error for the least squares estimate of γ, construct a 95% confidence interval for γ. Would null hypotheses of the form $H_0 : \gamma = 1$ and $H_0 : \gamma = 2$ be rejected? Comment.

$$cov(\hat{\beta}_{FGLS}) = \begin{bmatrix} 8.72441006608799 & -2.30350400891717 \\ -2.30350400891717 & 0.615304546386368 \end{bmatrix}$$

$$\begin{aligned} CI &: 15.1915 \pm 1.686\sqrt{0.615304546386368} \\ &: (140967, 16.7417) \end{aligned}$$

$H_0 : \gamma = 1$
$H_1 : \gamma \neq 1$

$$\begin{aligned} t &= \frac{15.1915 - 1}{\sqrt{0.615304546386368}} \\ &= 18.3822 \end{aligned}$$

$H_0 : \gamma = 2$
$H_1 : \gamma \neq 2$

$$\begin{aligned} t &= \frac{15.4192 - 2}{\sqrt{0.615304546386368}} \\ &= 17.1074 \end{aligned}$$

For both null hypotheses $H_0 : \gamma = 1$ and $H_0 : \gamma = 2$, we have sufficient evidence to reject them.

2.b.(iv)

Denote the least squares estimates of α and γ. Compute the variance estimates

$$\hat{\sigma}_t^2 = exp\{\hat{\alpha} + \hat{\gamma}\ln x_t\} = \begin{bmatrix} 8045.701 \\ 640758.5 \\ 17385607 \\ 75584362 \\ 1.25E+08 \\ 1.45E+08 \\ 1.91E+08 \\ 3.83E+08 \\ 7.67E+08 \\ 8.93E+08 \\ 1.39E+09 \\ 2.62E+09 \\ 2.83E+09 \\ 4.27E+09 \\ 4.71E+09 \\ 7.7E+09 \\ 1.53E+10 \\ 1.69E+10 \\ 4.19E+10 \\ 4.24E+10 \\ 5.87E+10 \\ 5.89E+10 \\ 6.19E+10 \\ 6.19E+10 \\ 8E+10 \\ 9.47E+10 \\ 1.05E+11 \\ 1.45E+11 \\ 3.64E+11 \\ 4.26E+11 \\ 4.86E+11 \\ 5.61E+11 \\ 5.68E+11 \\ 2.5E+12 \\ 2.5E+12 \\ 3.03E+12 \\ 4.35E+12 \\ 1.16E+13 \\ 7.16E+13 \\ 8.57E+13 \end{bmatrix}$$

2.b.(v)

Use the variance estimates obtained in part (iv) to find estimated generalized least squares estimates for β_1 and β_2. Report the results in the usual way. Based on your results in this part and part (a), and the results recorded in the text, do you think the estimates for β_1 and β_2, and their standard errors are very sensitive to the assumed form of heteroskedasticity?

$$\beta_{GLS} = \begin{bmatrix} -40.2498 \\ 15.1915 \end{bmatrix}$$

The estimates for β_1 and β_2 are not very sensitive to the assumed form of heteroskedasticity; however, their standard errors are highly sensitive.

3 §16.1

Suppose that a general linear statistical model has the AR(I) error model

$$e_t = \rho e_{t-1} + \epsilon_t$$

where $E[\epsilon_t] = 0$ for $t \neq s$ and $var(\epsilon_t) = E[\epsilon_t^2] = \sigma_\epsilon^2$ Suppose (unrealistically) that $T = 4$.

3.a

Using the notation in the text show that $E[ee'] = \sigma_\epsilon^2 V$ where

$$V = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix}$$

Since $T = 4$, our e_t should look like:

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

and

$$ee' = \begin{bmatrix} e_1^2 & e_1e_2 & e_1e_3 & e_1e_4 \\ e_2e_1 & e_2^2 & e_2e_3 & e_2e_4 \\ e_3e_1 & e_3e_2 & e_3^2 & e_3e_4 \\ e_4e_1 & e_4e_2 & e_4e_3 & e_{4^2} \end{bmatrix}$$

$$E(ee') = \begin{bmatrix} E(e_1^2) & E(e_1e_2) & \dots & E(e_1e_4) \\ \vdots & E(e_2^2) & \dots & E(e_2e_4) \\ \vdots & \vdots & \ddots & \vdots \\ E(e_4e_1) & \dots & \dots & E(e_4^2) \end{bmatrix}$$

$$E(ee') = W = \begin{bmatrix} var(e_1) & cov(e_1,e_2) & \dots & cov(e_1,e_4) \\ cov(e_2,e_1) & var(e_2) & \dots & cov(e_2,e_4) \\ & & \ddots & \\ cov(e_3,e_1) & \dots & & cov(e_3,e_4) \\ cov(e_4,e_1) & \dots & \dots & var(e_4) \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma^2\rho & \sigma^2\rho^2 & \sigma^2\rho^3 \\ \sigma^2\rho & \sigma_2^2 & \sigma^2\rho & \sigma^2\rho^2 \\ \sigma^2\rho^2 & \sigma^2\rho & \sigma_3^2 & \sigma^2\rho \\ \sigma^2\rho^3 & \sigma^2\rho^2 & \sigma^2\rho & \sigma_4^2 \end{bmatrix} \tag{2}$$

$$= \sigma_e^2 \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix} \tag{3}$$

$$var(e_t) = \rho^2 var(e_{t-1}) + var(\epsilon_t) + 2\rho cov(e_{t-1}, \epsilon_t) \tag{1}$$

$$= \rho^2 var(e_t) + \sigma_\epsilon^2 + 0 \tag{2}$$

$$= (1-\rho^2)\sigma_\epsilon^2 = \sigma_\epsilon^2 \tag{3}$$

$$\sigma_e^2 = \frac{\sigma_\epsilon^2}{1-\rho^2} \tag{4}$$

$$E(ee') = \sigma_e^2 \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix} \tag{1}$$

$$= \frac{\sigma_\epsilon^2}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix} \tag{2}$$

$$= \sigma_\epsilon^2 \frac{1}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix} \tag{3}$$

$$= \sigma_\epsilon^2 V \tag{4}$$

3.b

Show that $VV^{-1} = I$ where

$$V^{-1} = \begin{bmatrix} 1 & -\rho & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & -\rho & 1 \end{bmatrix}$$

$$VV^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & \rho & \rho^2 & \rho^3 \\ \rho & 1 & \rho & \rho^2 \\ \rho^2 & \rho & 1 & \rho \\ \rho^3 & \rho^2 & \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & -\rho & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & -\rho & 1 \end{bmatrix} \tag{1}$$

$$= \frac{1}{1-\rho^2} \begin{bmatrix} 1-\rho^2 & -\rho+\rho(1+\rho^2)-\rho^3 & -\rho^2+\rho^2(1+\rho^2)-\rho^4 & -\rho^3+\rho^3 \\ \rho-\rho & -\rho^2+(1+\rho^2)-\rho^2 & -\rho+\rho(1+\rho^2)-\rho^3 & -\rho^2+\rho^2 \\ \rho^2-\rho^2 & -\rho^3+\rho(1+\rho^2)-\rho^2 & -\rho^2+(1+\rho^2)-\rho^2 & -\rho+\rho \\ \rho^3-\rho^3 & -\rho^4+\rho^2(1+\rho^2)-\rho^2 & \rho^3+\rho(1+\rho^2)-\rho & -\rho^2+1 \end{bmatrix} \tag{2}$$

$$= \frac{1}{1-\rho^2} \begin{bmatrix} 1-\rho^2 & 0 & 0 & 0 \\ 0 & 1-\rho^2 & 0 & 0 \\ 0 & 0 & 1-\rho^2 & 0 \\ 0 & 0 & 0 & 1-\rho^2 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$\tag{5}$$

$$= I \tag{6}$$

$$VV^{-1} = I \tag{7}$$

3.c

Show that $P'P = V^{-1}$ where

$$P = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & 0 \\ -\rho & 1 & 0 & 0 \\ 0 & -\rho & 1 & 0 \\ 0 & 0 & -\rho & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} \sqrt{1-\rho^2} & -\rho & 0 & 0 \\ 0 & 1 & -\rho & 0 \\ 0 & 0 & 1 & -\rho \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P'P = \begin{bmatrix} \sqrt{1-\rho^2} & -\rho & 0 & 0 \\ 0 & 1 & -\rho & 0 \\ 0 & 0 & 1 & -\rho \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & 0 \\ -\rho & 1 & 0 & 0 \\ 0 & -\rho & 1 & 0 \\ 0 & 0 & -\rho & 1 \end{bmatrix} \quad (1)$$

$$= \begin{bmatrix} \sqrt{1-\rho^2}\sqrt{1-\rho^2}+\rho^2 & -\rho & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & -\rho & 1 \end{bmatrix} \quad (2)$$

$$= \begin{bmatrix} 1 & -\rho & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho \\ 0 & 0 & -\rho & 1 \end{bmatrix} \quad (3)$$

$$P'P = V^{-1} \quad (4)$$

3.d

Let $y* = Py$ where

$$y* = \begin{bmatrix} y_1* \\ y_2* \\ y_3* \\ y_4* \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Find each of the y_t* in terms of the y_t.

$$y* = Py$$

$$= \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & 0 \\ -\rho & 1 & 0 & 0 \\ 0 & -\rho & 1 & 0 \\ 0 & 0 & -\rho & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= \begin{bmatrix} y_1\sqrt{1-\rho^2} \\ -y_1\rho + y_2 \\ -y_2\rho + y_3 \\ -y_3\rho + y_4 \end{bmatrix}$$

Therefore, $y_1* = y_1\sqrt{1-\rho^2}$, $y_2* = -y_1\rho + y_2$, $y_3* = -y_2\rho + y_3$, and $y_4* = -y_3\rho + y_4$

3.e

Explain why the results in parts (a), (b), (c), and (d) imply that the generalized least squares estimator $\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}y$ can be computed using the transformations in equation 16.7.11.

Parts (a), (b), (c), and (d) have resulted in correcting the existence of autocorrelated errors. As we can see, with the inclusion of V^{-1} into our OLS formula into a generalized least squares estimator, we have included an autoregressive process that includes the possibility that the error term in our data are related. Hence, we see from the previous parts y_t* is affected by $\rho y_t - y_{t-1}$, where $-1 < \rho < 1$. This allows us to check for the presence of autocorrelated errors in any time-series data.